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THE FREQUENCY SPECTRUM IN PHASE-PULSE MODULATION

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[Figures referred to herein are appended.]

In this article a theoretical analysis is made of the frequency spectrum obtained in a fixed cycle at the output of a transmitter operating on a phase-pulse modulation system. It is demonstrated that the frequency spectrum possesses a complicated structure: it consists of an infinite series of fundamental overtones, short pulses of mean frequency, and infinite series of cumulative and differential combination frequencies (between mean pulse frequency and modulating frequency), accompanying the fundamental overtones. Expressions are given for the amplitude and phase of each frequency in the spectrum. At the end of the article a numerical example is considered.

Nowadays, for the most part, a considerable number of transmissions are carried on simultaneously over the channel of communication used for telemechanics, communications, etc. An extremely widespread method is that in which different frequency bands, filtered on reception, are used. In other cases a communication channel is assigned, by means of a transmitting and receiving apparatus, to each subscriber in turn for very short intervals of time. Often the subscribers not only do not detect this circumstance, but do not even notice the presence of other parties on the same channel.

There are excellent prospects for the development of a second method of using communication channels, particularly in view of new possibilities which have arisen in connection with electronic commutators and with ultrashort wave, decimeter and even shorter wave radio-lines; the latter is of importance mainly for telephony, phototelegraphy, etc.

Alternate use of a communication channel by subscribers may be effected by several methods. First, alternate assignment of a communication channel may be made for such short intervals of time, and also so often, that the period of the alternating currents of the transmitted signals (of telemechanics, telephony, etc.) is greater than the time intervals. Such a method is known as pulse transmission.

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Second, the alternate assignment of a communication channel to subscribers can be effected for such intervals of time that, during each of them, it is possible to transmit several waves of alternating current signals.

As each communication channel has its own inherent characteristics, (frequency, phase), in order to detect distortions arising during the use of such an actual communication channel, and also mutual interference between various subscribers using this channel, it is first necessary to know the frequency spectrum appearing in the transmitting apparatus and the amplitudes and phases of these frequencies.

The basic types of pulse modulation are shown in Figure 1. Here t is time and E_1 is the voltage supplied by subscriber No 1 for modulation of the transmitting installation. For greater clarity of the diagram, it is assumed that in the given case the transmitting installation is designed for only two subscribers, and that in the interval of time t_1 the transmitting installation is assigned to subscriber No 1, and in the interval of time t_2 to subscriber No 2. Figure 1 shows the pulses received at the output of the transmitting installation during transmission by subscriber No 1 only. In amplitude-pulse modulation (coordinates E_a -- t) the duration ξ_a of each pulse and the interval of time θ_a between the centers of two adjacent pulses remain constant and the amplitude e_a of each pulse is proportional to the momentary value of the voltage e_1 of the transmitted signal. In time-pulse modulation (coordinates E_s -- t), the amplitude e_s of each pulse and the interval of time θ_s between the centers of two adjacent pulses remain constant, and the duration ξ_s of each pulse is proportional to the momentary value of the voltage e_1 of the transmitted signal. In phase-pulse modulation (coordinates E_t -- t) the amplitude e_t and the duration ξ_t of each pulse remain constant, and the time θ_t of the advance or lag of each pulse is a function of the momentary value e_1 of the voltage (and sign) of the transmitted signal. A description of the technical methods of effecting this or that type of pulse modulation will not be included in this article. The literature on problems of pulse communication is extensive; we shall cite a few articles [1-5].

The purpose of this article is to study the frequency spectrum obtained at the output of a transmitting station operating on a phase-pulse modulation system. For the sake of simplicity, we shall assume that only one subscriber, out of the total number of subscribers N, controls the transmission, and that the remaining subscribers N-1 do not transmit. Let $\zeta(t)$ be a complex signal transmitted by this subscriber. Let us also assume that the function $\zeta(t)$ is broken down into a series of sine components; let us limit our examination to the conditions obtaining during the transmission on a fixed cycle of only one such simple sine component forming part of a complicated signal. Let the equation of this simple wave be

$$e = E_0 + e_0 \sin\left(\frac{2\pi t}{\tau} + \beta\right), \tag{1}$$

where τ (seconds) is the period of the wave, e_0 (volts) its amplitude, E_0 (volts) the constant component, and β the phase of the wave.

Let us denote by P the average number of pulses transmitted in one second, that is, the mean pulse frequency. It is evident that during the fixed cycle each wave (1) will be broken up into $P\tau$ pulses. This number, generally speaking, may be any number--whole, fractional, irrational, etc. Let us limit our examination further to cases which are quite sufficient for a practical solution to the problem before us: when $P\tau$ is an integer or a fractional number. If $P\tau$ is an integer, the picture of the pulse transmission of any wave will in no way differ from the picture of the pulse transmission of previous and successive waves: in this case τ is the period after which the picture is exactly repeated.

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In the other case, $P\tau$ is, on the average, a fractional number, and therefore the picture of pulse transmission of each sine wave with a period τ (seconds) will differ somewhat from the picture of the previous and successive waves. In this second, more general case for the period after which the picture is exactly repeated, it is necessary to take a greater time interval T (seconds) which for an integer is once again as large as both the period τ of the transmitted sine waves and the average time interval $\frac{1}{P}$ (seconds) of the recurrence of the pulses.

Let us denote by ν (cycles) the frequency of the transmitted sine wave. It is clear that

$$\nu\tau = 1. \tag{2}$$

Let us introduce the notation

$$T = \alpha\tau. \tag{3}$$

According to agreement, α is an integer. The particular case just mentioned ($P\tau$ being an integer) corresponds with the $\alpha = 1$.

Figure 2 shows the distribution of transmissions between individual subscribers during the time interval T. Here the time intervals during which the transmitter is put at the disposition of subscriber No 1 are crosshatched. It is evident that each pulse is located somewhere within the crosshatched area, and that its position is dependent on the momentary value e of the voltage of the signal transmitted by this subscriber, indicated in Figure 2 by a continuous curve.

Figure 3 illustrates on an enlarged scale the segment of time corresponding to the crosshatched area b in Figure 2. At the moment $\frac{n}{P}$ and until the moment $\frac{n}{P} + \frac{1}{PN} = \frac{nN+1}{PN}$, the transmitter is at the disposition of subscriber No 1. If subscriber No 1 made no transmission at all, the center of the pulse would coincide with the center of the time interval assigned to this subscriber, i.e., it would correspond to the moment

$$\sigma_{n+1} = \frac{2nN+1}{2PN}. \tag{4}$$

It is evident that n varies within the limits

$$0 \leqslant n \leqslant PT-1. \tag{5}$$

However, at moment (4) the momentary voltage transmitted by subscriber No 1, in accordance with (1), will be

$$e_{n+1} = E_0 + e_0 \sin\left[\frac{(2nN+1)\pi}{PN\tau} + \beta\right]. \tag{6}$$

As a result of this, pulse number n + 1 will be transmitted later, at θ_{n+1} (seconds), and

$$\theta_{n+1} = \varepsilon e_{n+1}, \tag{7}$$

where ε is the factor of proportionality.

The duration ξ (seconds) of a pulse and its amplitude E (volts) do not depend on the pulse cardinal number n.

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Let us denote by S the relation of the time interval $\frac{1}{PN}$, during which the transmitter is at the disposition of each subscriber, to the pulse duration ξ, i.e.,

$$S = \frac{1}{PN\xi}. \tag{8}$$

It may be seen from Figure 3 that the greatest possible value of θ must satisfy the condition

$$\theta_{max} = \frac{1}{2PN} - \frac{\xi}{2},$$

or, in accordance with (8),

$$\theta_{max} = \frac{S-1}{2PNS}.$$

But as it is evident on the basis of (7) and (1) that

$$\theta_{max} = \varepsilon E_0 + \varepsilon e_0,$$

it follows that

$$\varepsilon E_0 + \varepsilon e_0 = \frac{S-1}{2PNS}.$$

Let us denote by μ_e the fractional modulation of the variable component and by μ_E, part of the largest possible value of the constant component; obviously

$$0 \leq \mu_e + \mu_e \leq 1,$$

and

$$0 \leq \mu_e \leq 1,$$

$$0 \leq \mu_E \leq 1$$

Thus, we can write expression (7) in the form

$$\theta_{n+1} = \frac{S-1}{2PNS}\left\{\mu_E + \mu_e \sin\left[\frac{(2nN+1)\pi}{PN\tau} + \beta\right]\right\}. \tag{9}$$

It is easy to see from Figure 3 that during modulation the center of a pulse will correspond with the moment

$$t_{n+1} = \sigma_{n+1} + \theta_{n+1},$$

or, in accordance with (4) and (9),

$$t_{n+1} = \frac{2nN+1}{2PN} + \frac{S-1}{2PNS}\left\{\mu_E + \mu_e \sin\left[\frac{(2nN+1)\pi}{PN\tau} + \beta\right]\right\}. \tag{10}$$

The moment t'_{n+1} of the beginning of the pulse will evidently be

$$t'_{n+1} = t_{n+1} - \frac{\xi}{2},$$

or, in accordance with (8),

$$t'_{n+1} = t_{n+1} - \frac{1}{2PNS}. \tag{11}$$

Similarly, the moment t''_{n+1} of the end of the pulse will be

$$t''_{n+1} = t_{n+1} + \frac{1}{2PNS}. \tag{12}$$

[redacted]

The problem of determining the frequency spectrum obtained at the output of a transmitter amounts to breaking down into a Fourier series the pulses shown in Figure 1 (coordinates E_t -- t). The breakdown into a Fourier series of pulses obtained in the absense of modulation may be represented by sufficiently simple, well-known relations [6-7]. Various authors have been interested in questions on the analysis of pulse transmission. Thus, for example, it is known [8] that this question was studied by Carson (1920) and Nyquist (1936), but their works were not published. It is likewise known that other authors [9] produced very brief fragmentary data touching the problem which interests us. The extremely great value of pulse transmission requires a detailed and exact examination of the problem formulated and attainment of the necessary relations in a form which will permit their immediate practical use leading to a numerical result.

The curve which we are about to study, as shown in Figures 2 and 3, has 2PT points of interruption and consists of rectilinear segments parallel to the axis of the abscissas or coinciding with the axis of the abscissas.

Thus, if F (t) is the function we are expanding into a Fourier series, we may write [10]

$$F(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos\frac{2k\pi t}{T}+\sum_{k=1}^{\infty} b_k \sin\frac{2k\pi t}{T}. \qquad (13)$$

The coefficients of the Fourier series are determined, as usual, by the relations

$$a_0=\frac{2}{T}\int_0^T F(t)\,dt, \qquad (14)$$

$$a_k=\frac{2}{T}\int_0^T F(t)\cos\frac{2k\pi t}{T}\,dt, \qquad (15)$$

or

$$b_k=\frac{2}{T}\int_0^T F(t)\sin\frac{2k\pi t}{T}\,dt. \qquad (16)$$

Here k is a positive integer receiving a value from one to infinity, i.e.,

$$1\leqslant k\leqslant\infty. \qquad (17)$$

The function F(t), in accordance with Figures 2 and 3 and (11) and (12), equals E in the time intervals

$$t_{n+1}-\frac{1}{2PNS}\leqslant t\leqslant t_{n+1}+\frac{1}{2PNS} \qquad (18)$$

and equals zero during the remaining intervals of time. Here n has integral values (5).

Let us begin by calculating the integral (14). Taking into consideration the properties of the function F (t), in accordance with (18), we have:

$$a_0=\frac{2}{T}\sum_{n=0}^{PT-1}\int_{t_{n+1}-\frac{1}{2PNS}}^{t_{n+1}+\frac{1}{2PNS}} E\,dt,$$

or

$$a_0=\frac{2E}{NS}. \qquad (19)$$

Let us calculate the second integral (15). Taking into consideration the properties of the function F(t), in accordance with (18), we have

$$a_k = \frac{2}{T} \sum_{n=0}^{PT-1} \int_{t_{n+1} - \frac{1}{2PNS}}^{t_{n+1} + \frac{1}{2PNS}} E \cos \frac{2k\pi t}{T} \, dt,$$

or

$$a_k = \frac{E}{k\pi} \sum_{n=0}^{PT-1} \left[\sin\left(\frac{2k\pi t_{n+1}}{T} + \frac{k\pi}{PNST} \right) - \sin\left(\frac{2k\pi t_{n+1}}{T} - \frac{k\pi}{PNST} \right) \right],$$

or

$$a_k = \frac{2E}{k\pi} \sin \frac{k\pi}{PNST} \sum_{n=0}^{PT-1} \cos \frac{2k\pi t_{n+1}}{T}. \qquad (20)$$

Finally, let us calculate the last integral (16). Again taking into consideration the properties of the function F(t), we shall obtain, in accordance with (18),

$$b_k = \frac{2}{T} \sum_{n=0}^{PT-1} \int_{t_{n+1} - \frac{1}{2PNS}}^{t_{n+1} + \frac{1}{2PNS}} E \sin \frac{2k\pi t}{T} \, dt,$$

or

$$b_k = -\frac{E}{k\pi} \sum_{n=0}^{PT-1} \left[\cos\left(\frac{2k\pi t_{n+1}}{T} + \frac{k\pi}{PNST} \right) - \cos\left(\frac{2k\pi t_{n+1}}{T} - \frac{k\pi}{PNST} \right) \right],$$

or

$$b_k = \frac{2E}{k\pi} \sin \frac{k\pi}{PNST} \sum_{n=0}^{PT-1} \sin \frac{2k\pi t_{n+1}}{T}. \qquad (21)$$

Now, taking into consideration (19), (20), and (21), we can write the expression (13), which interests us, in the form

$$F(t) = \frac{E}{NS} + \sum_{k=1}^{\infty} \frac{2E}{k\pi} \sin \frac{k\pi}{PNST} \cos \frac{2k\pi t}{T} \sum_{n=0}^{PT-1} \cos \frac{2k\pi t_{n+1}}{T} +$$

$$+ \sum_{k=1}^{\infty} \frac{2E}{k\pi} \sin \frac{k\pi}{PNST} \sin \frac{2k\pi t}{T} \sum_{n=0}^{PT-1} \sin \frac{2k\pi t_{n+1}}{T}, \qquad (22)$$

or (see Appendix)

$$F(t) = \frac{E}{NS} + \sum_{u=1}^{\infty} \sum_{m=0}^{\infty} \frac{2EP}{(uP + m\nu)\pi} J_m \left[\frac{\mu_E (S-1)(uP + m\nu)\pi}{PNS} \right] \sin \frac{(uP + m\nu)\pi}{PNS} \times$$

$$\times \cos \left[2(uP + m\nu)\pi t + m\pi - \frac{u\pi}{N} + m\beta - \frac{\mu_E (S-1)(uP + m\nu)\pi}{PNS} \right] +$$

$$+ \sum_{v=0}^{\infty} \sum_{m=1}^{\infty} \frac{2EP}{(vP - m\nu)\pi} J_m \left[\frac{\mu_E (S-1)(vP - m\nu)\pi}{PNS} \right] \frac{\sin (vP - m\nu)\pi}{PNS} \times$$

$$\times \cos \left[2(vP - m\nu)\pi t - \frac{v\pi}{N} - m\beta - \frac{\mu_E (S-1)(vP - m\nu)\pi}{PNS} \right]. \qquad (23)$$

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Such is the final expression of the breakdown into a Fourier series. We see that a frequency spectrum during phase-pulse modulation has a complex structure. Let us analyze the result obtained.

Let us first observe that the magnitudes T and α do not enter into our breakdown (23). This means that the frequency spectrum obtained does not depend on whether or not the period τ of a modulated wave is a multiple of the average pulse period $\frac{1}{P}$.

In the frequency spectrum obtained there is a constant component determined by the first member of the right-hand side of formula (23). The magnitude of the constant component is

$$a = \frac{E}{NS}. \qquad (24)$$

In the spectrum there are cumulative, combination components forming a dual series for which the frequencies, amplitudes and phases will accordingly be

$$\nu_{u+m} = uP + m\nu, \qquad (25)$$

$$a_{u+m} = \left| \frac{2EP}{(uP+m\nu)\pi} J_m\left[\frac{\mu_E(S-1)(uP+m\nu)\pi}{PNS}\right] \sin\frac{(uP+m\nu)\pi}{PNS} \right|, \qquad (26)$$

$$\varphi_{u+m} = m\pi - \frac{u\pi}{N} + m\beta - \frac{\mu_E(S-1)(uP+m\nu)\pi}{PNS}. \qquad (27)$$

Here the ordinal number u may be any positive integer from one to infinity, and the ordinal number m, zero or any positive integer from one to infinity. The ordinal numbers u and m can enter into any combination with each other.

As m may equal zero, we see that there are in the spectrum components with the frequencies uP which are overtones of the pulse frequency P. Each pulse frequency overtone has an infinite number of cumulative side components (obtained when $m \neq 0$); these frequencies are located on the right of their fundamental overtone uP.

Formula (23) shows that in the spectrum there are also differential combination components, forming a dual series, for which the frequencies, amplitudes and phases will accordingly be

$$\nu_{v-m} = |vP - m\nu|, \qquad (28)$$

$$a_{v-m} = \left| \frac{2EP}{(vP-m\nu)\pi} J_m\left[\frac{\mu_e(S-1)(vP-m\nu)\pi}{PNS}\right] \sin\frac{(vP-m\nu)\pi}{PNS} \right|, \qquad (29)$$

$$\varphi_{v-m} = \pm\left[-\frac{v\pi}{N} - m\beta - \frac{\mu_F(S-1)(vP-m\nu)\pi}{PNS}\right]. \qquad (30)$$

Here the ordinal number v may be zero or any positive integer from one to infinity, and the ordinal number m may be any positive integer from one to infinity. The plus and minus signs in formula (30) refer to cases where v is greater or less, respectively, than $\frac{m\nu}{P}$. The ordinal numbers v and m may enter into any combination with each other.

As v may equal zero, we see that there are in the spectrum components with the frequencies $m\nu$, which are overtones of the modulating frequency ν. A fundamental tone is obtained when m = 1.

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We likewise see from formulas (28) and (29) that each frequency pulse overtone νP has an infinite number of differential side components; these frequencies are located to the left as well as to the right of their fundamental overtone νP.

Let us observe that cosine functions of the time enter into formula (23), whereas the modulating wave (1) is a sine function. This means that it is necessary to add the angle $\frac{\pi}{2}$ each time to the value of the phases φ_{u+m} and φ_{y-m}, calculated in accordance with formulas (27) and (30). Furthermore, in calculating the phase, the signs of amplitudes (26) and (29) must be verified. If the amplitude obtained is negative, it is clear that, in this case, the angle π must be added to the phase.

For a graphic representation of the frequency spectrum obtained and of the frequency amplitudes, let us take a numerical example.

The number of subscribers is

$$N = 6. \tag{31}$$

The ratio of the time interval, during which the transmitter is assigned to each subscriber, to the pulse duration is

$$S = 4. \tag{32}$$

The fractional modulation of the variable component is

$$\mu_e = 1. \tag{33}$$

We shall express the modulating frequency in terms of the average number of pulses per second

$$\nu = \frac{P}{3}. \tag{34}$$

Let us take the pulse amplitude as

$$E = 1200; \tag{35}$$

since in this case, at the numerical values (31), (32), and (33) the maximum value of amplitude (29) comes out equal to 100 (in the absense of modulating frequency and when the ordinal number of the fundamental overtone equals zero).

Figure 4 gives a graphic interpretation of the spectrum obtained in the given case for frequencies and their amplitudes. Finally, Figure 5 represents the spectrum of frequencies differing only in that the modulating frequency has a value not of (34), but equal to

$$\nu = \frac{P}{14}. \tag{36}$$

It is easy to see, on the basis of Figures 4 and 5, that side bands can be constructed quite simply if subscriber No 1 does not transmit just one frequency, (34) or (36), but a whole frequency band.

APPENDIX

Let us take relation (22) for the original expression, i.e.,

$$F(t) = \frac{E}{NS} + \sum_{k=1}^{\infty} \frac{2E}{k\pi} \sin\frac{k\pi}{PNST} \cos\frac{2k\pi t}{T} \sum_{n=0}^{PT-1} \cos\frac{2k\pi t_{n+1}}{T} + \sum_{k=1}^{\infty} \frac{2E}{k\pi} \sin\frac{k\pi}{PNST} \sin\frac{2k\pi t}{T} \sum_{n=0}^{PT-1} \sin\frac{2k\pi t_{n+1}}{T} \tag{37}$$

or

$$F(t) = \frac{E}{NS} + \sum_{k=1}^{\infty} \frac{2E}{k\pi} \sin\frac{k\pi}{PNST} \sum_{n=0}^{PT-1} \cos\left(\frac{2k\pi t}{T} - \frac{2k\pi t_{n+1}}{T}\right) \tag{38}$$

For further transformation of expression (38), let us introduce, for the sake of brevity, the notation

$$R = \sum_{n=0}^{PT-1} \cos\left(\frac{2k\pi t}{T} - \frac{2k\pi t_{n+1}}{T}\right), \tag{39}$$

or, taking (10) into account, we shall obtain

$$R = \sum_{n=0}^{PT-1} \cos\left\{\frac{(2nN+1)k\pi}{PNT} + \frac{\mu_E (S-1)k\pi}{PNST} + \frac{\mu_E (S-1)k\pi}{PNST}\sin\left[\frac{(2nN+1)\pi}{PN\tau} + \beta\right] - \frac{2k\pi t}{T}\right\}, \tag{40}$$

or

$$R = \sum_{n=0}^{PT-1} \cos[An + \psi + C\sin(Dn + \varphi)], \tag{41}$$

where the following designations are made for brevity:

$$A = \frac{2k\pi}{PT}, \tag{42}$$

$$\psi = \frac{k\pi}{PNT} + \frac{\mu_E (S-1)k\pi}{PNST} - \frac{2k\pi t}{T}, \tag{43}$$

$$C = \frac{\mu_E (S-1)k\pi}{PNST}, \tag{44}$$

$$D = \frac{2\pi}{P\tau}, \tag{45}$$

and

$$\varphi = \frac{\pi}{PN\tau} + \beta. \tag{46}$$

From (41) we shall have

$$R = \sum_{n=0}^{PT-1} \cos(An + \psi)\cos[C\sin(Dn + \varphi)] - \sum_{n=0}^{PT-1} \sin(An + \psi)\sin[C\sin(Dn + \varphi)]. \tag{47}$$

Let us use the well-known [11] relations between the trigonometric functions and the Bessel functions, which we shall here write in the form

$$\cos(C\sin\gamma) = J_0(C) + 2\sum_{m=1}^{\infty} J_{2m}(C)\cos 2m\gamma, \tag{48}$$

$$\sin(C\sin\gamma) = 2\sum_{m=1}^{\infty} J_{2m-1}(C)\sin[(2m-1)\gamma]. \tag{49}$$

Expression (47), on the basis of (48) and (49), will take the form

$$R = \sum_{n=0}^{PT-1} \cos(An + \psi) J_0(C) + \sum_{n=0}^{PT-1} \cos(An + \psi)\cdot 2\sum_{m=1}^{\infty} J_{2m}(C)\cos[2m(Dn + \varphi)] - \sum_{n=0}^{PT-1} \sin(An + \psi)\cdot 2\sum_{m=1}^{\infty} J_{2m-1}(C)\sin[(2m-1)(Dn + \varphi)], \tag{50}$$

or, changing the form of the addition, we shall obtain

$$
\begin{aligned}
R = {} & J_0(C)\sum_{n=0}^{PT-1}\cos(An+\psi) + \\
& + \sum_{m=1}^{\infty} J_{2m-1}(C)\sum_{n=0}^{PT-1}\cos\{[(2m-1)D+A]n+[(2m-1)\varphi+\psi]\} + \\
& + \sum_{m=1}^{\infty} J_{2m}(C)\sum_{n=0}^{PT-1}\cos\{[(2m)D+A]n+[(2m)\varphi+\psi]\} + \\
& + \sum_{m=1}^{\infty} (-1)^{2m-1}J_{2m-1}(C)\sum_{n=0}^{PT-1}\cos\{[(2m-1)D-A]n+[(2m-1)\varphi-\psi]\} + \\
& + \sum_{m=1}^{\infty} (-1)^{2m}J_{2m}(C)\sum_{n=0}^{PT-1}\cos\{[(2m)D-A]n+[(2m)\varphi-\psi]\}. \qquad (51)
\end{aligned}
$$

Let us examine the expressions in the second and third lines of formula (51). We shall see that these expressions differ only in that in the second line the addition is carried out for all odd values and in the third line for all even values. Thus, both expressions in the second and third lines may be replaced by one general expression. Exactly the same thing can be done with the expressions in the fourth and fifth lines. The expression in the first line may also be joined with the expressions in the second and third lines. Thus

$$
\begin{aligned}
R = {} & \sum_{m=0}^{\infty} J_m(C)\sum_{n=0}^{PT-1}\cos[(mD+A)n+(m\varphi+\psi)] + \\
& + \sum_{m=1}^{\infty}(-1)^m J_m(C)\sum_{n=0}^{PT-1}\cos[(mD-A)n+(m\varphi-\psi)].
\end{aligned} \qquad (52)
$$

Let us use a known [12] relation, which we shall write here in the form

$$
\sum_{n=0}^{PT-1}\cos(yn+\delta) = \frac{\sin\frac{PTy}{2}\cos\left[(PT-1)\frac{y}{2}+\delta\right]}{\sin\frac{y}{2}}. \qquad (53)
$$

Expression (52) with calculation (53) takes the form:

$$
\begin{aligned}
R = {} & \sum_{m=0}^{\infty} J_m(C)\frac{\sin\frac{PT(mD+A)}{2}\cos\left[\frac{(PT-1)(mD+A)}{2}+(m\varphi+\psi)\right]}{\sin\frac{mD+A}{2}} + \\
& + \sum_{m=1}^{\infty}(-1)^m J_m(C)\frac{\sin\frac{PT(mD-A)}{2}\cos\left[\frac{(PT-1)(mD-A)}{2}+(m\varphi-\psi)\right]}{\sin\frac{mD-A}{2}}
\end{aligned} \qquad (54)
$$

It follows from (45), (42), (46), and (43) that

$$
\frac{mD+A}{2} = \frac{m\pi}{P\tau} + \frac{k\pi}{PT}, \qquad (55)
$$

$$
\frac{mD-A}{2} = \frac{m\pi}{P\tau} - \frac{k\pi}{PT}, \qquad (56)
$$

$$
m\varphi+\psi = \frac{m\pi}{PN\tau} + m\beta + \frac{k\pi}{PNT} + \frac{\mu_E(S-1)k\pi}{PNST} - \frac{2k\pi t}{T} \qquad (57)
$$

and

$$
m\varphi-\psi = \frac{m\pi}{PN\tau} + m\beta - \frac{k\pi}{PNT} - \frac{\mu_E(S-1)k\pi}{PNST} + \frac{2k\pi t}{T}. \qquad (58)
$$

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On the basis of (44), (55), (56), (57) and (58), expression (54) takes the form

$$R=\sum_{m=0}^{\infty} J_m\left[\frac{\mu_e(S-1)k\pi}{PNST}\right]\sin\left(\frac{m\pi T}{\tau}+k\pi\right)\times$$

$$\times\frac{\cos\left[\frac{(PT-1)m\pi}{P\tau}+\frac{(PT-1)k\pi}{PT}+\frac{m\pi}{PN\tau}+m\beta+\frac{k\pi}{PNT}+\frac{\mu_E(S-1)k\pi}{PNST}-\frac{2k\pi t}{T}\right]}{\sin\left(\frac{m\pi}{P\tau}+\frac{k\pi}{PT}\right)}+$$

$$+\sum_{m=1}^{\infty}(-1)^m J_m\left[\frac{\mu_e(S-1)k\pi}{PNST}\right]\sin\left(\frac{m\pi T}{\tau}-k\pi\right)\times$$

$$\times\frac{\cos\left[\frac{(PT-1)m\pi}{P\tau}-\frac{(PT-1)k\pi}{PT}-\frac{m\pi}{PN\tau}+m\beta-\frac{k\pi}{PNT}-\frac{\mu_E(S-1)k\pi}{PNST}+\frac{2k\pi t}{T}\right]}{\sin\left(\frac{m\pi}{P\tau}-\frac{k\pi}{PT}\right)} \qquad (59)$$

On the basis of (39), (59), and (3), expression (38), which is of interest to us, can now be written in the form

$$F(t)=\frac{E}{NS}+\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}2E\sin[(m\alpha+k)\pi]\sin\frac{k\pi}{PNS\alpha\tau}J_m\left[\frac{\mu_e(S-1)k\pi}{PNS\alpha\tau}\right]\times$$

$$\times\frac{\cos\left[\frac{2k\pi t}{\alpha\tau}-\frac{(P\alpha\tau-1)(m\alpha+k)\pi}{P\alpha\tau}-\frac{(m\alpha+k)\pi}{PN\alpha\tau}+m\beta-\frac{\mu_E(S-1)k\pi}{PNS\alpha\tau}\right]}{k\pi\sin\frac{(m\alpha+k)\pi}{P\alpha\tau}}+$$

$$+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}2E\sin[(m\alpha-k)\pi]\sin\frac{k\pi}{PNS\alpha\tau}J_m\left[\frac{\mu_e(S-1)k\pi}{PNS\alpha\tau}\right]\times$$

$$\times\frac{\cos\left[\frac{2k\pi t}{\alpha\tau}+\frac{(P\alpha\tau-1)(m\alpha-k)\pi}{P\alpha\tau}+\frac{(m\alpha-k)\pi}{PN\alpha\tau}+m\beta-\frac{\mu_E(S-1)k\pi}{PNS\alpha\tau}\right]}{(-1)^m k\pi\sin\frac{(m\alpha-k)\pi}{P\alpha\tau}} \qquad (60)$$

As m, α and k are integers, we see that the numerators of all the expressions under the sign of a double sum are always reduced to zero. Thus, as a rule, members of a double sum reduce to zero; exception may be made only of those members in which the denominator is reduced to zero at the same time. Such members become indeterminate, of the form $\frac{0}{0}$. It is therefore necessary to study them further. Let us write expression (60) in the form

$$F(t)=\frac{E}{NS}+\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}G_1(k,m)+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}G_2(k,m), \qquad (61)$$

where for the sake of brevity the following notation is used:

$$G_1(k,m)=2E\sin[(m\alpha+k)\pi]\sin\frac{k\pi}{PNS\alpha\tau}J_m\left[\frac{\mu_e(S-1)k\pi}{PNS\alpha\tau}\right]\times$$

$$\times\frac{\cos\left[\frac{2k\pi t}{\alpha\tau}-\frac{(P\alpha\tau-1)(m\alpha+k)\pi}{P\alpha\tau}-\frac{(m\alpha+k)\pi}{PN\alpha\tau}-m\beta-\frac{\mu_E(S-1)k\pi}{PNS\alpha\tau}\right]}{k\pi\sin\frac{(m\alpha+k)\pi}{P\alpha\tau}}, \qquad (62)$$

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$$G_2(k,m)=2E\sin[(m\alpha-k)\pi]\sin\frac{k\pi}{PNS\alpha\tau}J_m\left[\frac{\mu_e(S-1)k\pi}{PNS\alpha\tau}\right]\times$$

$$\times\frac{\cos\left[\frac{2k\pi t}{\alpha\tau}+\frac{(P\alpha\tau-1)(m\alpha-k)\pi}{P\alpha\tau}+\frac{(m\alpha-k)\pi}{PN\alpha\tau}+m\beta-\frac{\mu_E(S-1)k\pi}{PNS\alpha\tau}\right]}{(-1)^m k\pi\sin\frac{(m\alpha-k)\pi}{P\alpha\tau}}. \qquad (63)$$

Let us begin by studying expression (62). We see that the denominator of this expression reduces to zero, if

$$\frac{m\alpha+k}{P\alpha\tau}=q, \qquad (64)$$

whence

$$k=\alpha(qP\tau-m). \qquad (65)$$

here, according to agreement, q must be an integer (positive or negative) or zero.

As k, α, $P\alpha\tau$ and m are positive integers (and, in addition, m may also be equal to zero), it follows from (65) that $qP\tau$ - m must be a positive number, and this will be true if

$$\frac{m}{P\tau}<q\leq\infty. \qquad (66)$$

Whence it follows that q can only be a positive integer greater than $\frac{m}{P\tau}$.

The indeterminatenesc of expression (62) under condition (65) may be shown by the usual method

$$\lim_{k\to\alpha(qP\tau-m)}G_1(k,m)=\lim_{k\to\alpha(qP\tau-m)}\left(2E\pi\cos[(m\alpha+k)\pi]\sin\frac{k\pi}{PNS\alpha\tau}J_m\left[\frac{\mu_e(S-1)k\pi}{PNS\alpha\tau}\right]\times\right.$$

$$\times\frac{\cos\left[\frac{}{\alpha\tau}-\frac{(P\alpha\tau-1)(m\alpha+k)\pi}{P\alpha\tau}-\frac{(m\alpha+k)\pi}{PN\alpha\tau}-m\beta-\frac{\mu_E(S-1)k\pi}{PNS\alpha\tau}\right]}{\pi\sin\frac{(m\alpha+k)\pi}{P\alpha\tau}+\frac{k\pi^2}{P\alpha\tau}\cos\frac{(m\alpha+k)\pi}{P\alpha\tau}}+$$

$$+\frac{2E\sin[(m\alpha+k)\pi]}{\pi\sin\frac{(m\alpha+k)\pi}{P\alpha\tau}+\frac{k\pi^2}{P\alpha\tau}\cos\frac{(m\alpha+k)\pi}{P\alpha\tau}}\cdot\frac{d}{dk}\left\{\sin\frac{k\pi}{PNS\alpha\tau}J_m\left[\frac{\mu_e(S-1)k\pi}{FNS\alpha\tau}\right]\times\right.$$

$$\left.\left.\times\cos\left[\frac{2k\pi t}{\alpha\tau}-\frac{(P\alpha\tau-1)(m\alpha+k)\pi}{P\alpha\tau}-\frac{(m\alpha+k)\pi}{PN\alpha\tau}-m\beta-\frac{\mu_E(S-1)k\pi}{PNS\alpha\tau}\right]\right\}\right) \qquad (67)$$

The second component always reduces to zero; the first component, regardless of whether the integers $qP\alpha\tau$ and q are even or odd, takes the form:

$$\lim_{k\to\alpha(qP\tau-m)}G_1(k,m)=\frac{2EP}{(qP-m\nu)\pi}\sin\frac{(qP-m\nu)\pi}{PNS}J_m\left[\frac{\mu_e(S-1)(qP-m\nu)\pi}{PNS}\right]\times$$

$$\times\cos\left[2(qP-m\nu)\pi t-\frac{q\pi}{N}-m\beta-\frac{\mu_E(S-1)(qP-m\nu)\pi}{PNS}\right]. \qquad (68)$$

Let us proceed to study the second expression (63). We see that the denominator of this expression reduces to zero if

$$\frac{m\alpha - k}{P\alpha\tau} = r, \tag{69}$$

or

$$k = \alpha(m - rP\tau). \tag{70}$$

Here, by agreement, r must be an integer (positive or negative) or zero.

As k, α, m and $P\alpha\tau$ are positive integers, it follows from (70) that $m - rP\tau$ must be a positive number, and this will be true if

$$\frac{m}{P\tau} > r \geq -\infty. \tag{71}$$

Whence it follows that r can be any negative integer, zero, and any positive integer less than $\frac{m}{P\tau}$.

The indeterminateness of expression (63) under condition (70) may be shown by the usual method:

$$\lim_{k\to\alpha(m-rP\tau)} G_1(k,m) = \lim_{k\to\alpha(m-rP\tau)} \Bigg(-2E\pi\cos[(m\alpha-k)\pi]\sin\frac{k\pi}{PNS\alpha\tau} J_m\left[\frac{\mu_E(S-1)k\pi}{PNS\alpha\tau}\right] \times$$
$$\times \frac{\cos\left[\frac{2k\pi t}{\alpha\tau} + \frac{(P\alpha\tau-1)(m\alpha-k)\pi}{P\alpha\tau} + \frac{(m\alpha-k)\pi}{PN\alpha\tau} + m\beta - \frac{\mu_E(S-1)k\pi}{PNS\alpha\tau}\right]}{(-1)^m\pi\sin\frac{(m\alpha-k)\pi}{P\alpha\tau} - (-1)^m\frac{k\pi^2}{P\alpha\tau}\cos\frac{(m\alpha-k)\pi}{P\alpha\tau}} +$$
$$+ \frac{2E\sin[(m\alpha-k)\pi]}{(-1)^m\pi\sin\frac{(m\alpha-k)\pi}{P\alpha\tau} - (-1)^m\frac{k\pi^2}{P\alpha\tau}\cos\frac{(m\alpha-k)\pi}{P\alpha\tau}} \times$$
$$\times \frac{d}{dk}\Bigg\{\sin\frac{k\pi}{PNS\alpha\tau} J_m\left[\frac{\mu_E(S-1)k\pi}{PNS\alpha\tau}\right] \times$$
$$\times \cos\left[\frac{2\pi kt}{\alpha\tau} + \frac{(P\alpha\tau-1)(m\alpha-k)\pi}{P\alpha\tau} + \frac{(m\alpha-k)\pi}{PN\alpha\tau} + m\beta - \frac{\mu_E(S-1)k\pi}{PNS\alpha\tau}\right]\Bigg\}\Bigg), \tag{72}$$

Whence, by analogy with the foregoing, we find:

$$\lim_{k\to\alpha(m-rP\tau)} G_2(k,m) = \frac{2EP}{(m\nu-rP)\pi}\sin\frac{(m\nu-rP)\pi}{PNS} J_m\left[\frac{\mu_E(S-1)(m\nu-rP)\pi}{PNS}\right] \times$$
$$\times \cos\left[2(m\nu-rP)\pi t + \frac{r\pi}{N} + m\beta + m\pi - \frac{\mu_E(S-1)(m\nu-rP)\pi}{PNS}\right]. \tag{73}$$

Let us now put in expression (61), instead of $G_1(k,m)$, and $G_2(k,m)$, their values (68) and (73); taking into account likewise (66), (71) and (2), we can write

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$$F(t)=\frac{E}{NS}+\sum_{q>\frac{m\nu}{P}}^{\infty}\sum_{m=1}^{\infty}\frac{2EP}{(qP-m\nu)\pi}J_m\left[\frac{\mu_E(S-1)(qP-m\nu)\pi}{PNS}\right]\times$$
$$\times\sin\frac{(qP-m\nu)\pi}{PNS}\cos\left[2(qP-m\nu)\pi t-\frac{q\pi}{N}-m\beta-\frac{\mu_E(S-1)(qP-m\nu)\pi}{PNS}\right]+$$
$$+\sum_{-\infty}^{r<\frac{m\nu}{P}}\sum_{m=1}^{\infty}\frac{2EP}{(m\nu-rP)\pi}J_m\left[\frac{\mu_E(S-1)(m\nu-rP)\pi}{PNS}\right]\sin\frac{(m\nu-rP)\pi}{PNS}\times$$
$$\times\cos\left[2(m\nu-rP)\pi t+\frac{r\pi}{N}+m\beta+m\pi-\frac{\mu_E(S-1)(m\nu-rP)\pi}{PNS}\right],$$

or

$$F(t)=\frac{E}{NS}+\sum_{r=-\infty}^{-1}\sum_{m=1}^{\infty}\frac{2EP}{(m\nu-rP)\pi}J_m\left[\frac{\mu_E(S-1)(m\nu-rP)\pi}{PNS}\right]\times$$
$$\times\sin\frac{(m\nu-rP)\pi}{PNS}\cos\left[2(m\nu-rP)\pi t+\frac{r\pi}{N}+m\beta+m\pi-\frac{\mu_E(S-1)(m\nu-rP)\pi}{PNS}\right]+$$
$$+\sum_{0}^{r<\frac{m\nu}{P}}\sum_{m=1}^{\infty}\frac{2EP}{(m\nu-rP)\pi}J_m\left[\frac{\mu_E(S-1)(m\nu-rP)\pi}{PNS}\right]\sin\frac{(m\nu-rP)\pi}{PNS}\times$$
$$\times\cos\left[2(m\nu-rP)\pi t+\frac{r\pi}{N}+m\beta+m\pi-\frac{\mu_E(S-1)(m\nu-rP)\pi}{PNS}\right]+$$
$$+\sum_{q>\frac{m\nu}{P}}^{\infty}\sum_{m=1}^{\infty}\frac{2EP}{(qP-m\nu)\pi}J_m\left[\frac{\mu_E(S-1)(qP-m\nu)\pi}{PNS}\right]\sin\frac{(qP-m\nu)\pi}{PNS}\times$$
$$\times\cos\left[2(qP-m\nu)\pi t-\frac{q\pi}{N}-m\beta-\frac{\mu_E(S-1)(qP-m\nu)\pi}{PNS}\right].$$

(74)

Let us introduce the notation

$$-r=u.$$

Taking this into consideration, let us rewrite expression (74) in this form

$$F(t)=\frac{E}{NS}+\sum_{q=1}^{\infty}\frac{2E}{q\pi}J_0\left[\frac{\mu_E(S-1)q\pi}{NS}\right]\sin\frac{q\pi}{NS}\times$$
$$\times\cos\left[2qP\pi t-\frac{q\pi}{N}-\frac{\mu_E(S-1)q\pi}{NS}\right]+$$
$$+\sum_{u=1}^{\infty}\sum_{m=1}^{\infty}\frac{2EP}{(uP+m\nu)\pi}J_m\left[\frac{\mu_E(S-1)(uP+m\nu)\pi}{PNS}\right]\sin\frac{(uP+m\nu)\pi}{PNS}\times$$
$$\times\cos\left[2(uP+m\nu)\pi t-\frac{u\pi}{N}+m\beta+m\pi-\frac{\mu_E(S-1)(uP+m\nu)\pi}{PNS}\right]+$$
$$+\sum_{0}^{r<\frac{m\nu}{P}}\sum_{m=1}^{\infty}\frac{2EP}{(m\nu-rP)\pi}(-1)^m J_m\left[\frac{\mu_E(S-1)(m\nu-rP)\pi}{PNS}\right]\sin\frac{(m\nu-rP)\pi}{PNS}\times$$
$$\times\cos\left[2(m\nu-rP)\pi t+\frac{r\pi}{N}+m\beta-\frac{\mu_E(S-1)(m\nu-rP)\pi}{PNS}\right]+$$
$$+\sum_{q>\frac{m\nu}{P}}^{\infty}\sum_{m=1}^{\infty}\frac{2EP}{(qP-m\nu)\pi}J_m\left[\frac{\mu_E(S-1)(qP-m\nu)\pi}{PNS}\right]\sin\frac{(qP-m\nu)\pi}{PNS}\times$$
$$\times\cos\left[2(qP-m\nu)\pi t-\frac{q\pi}{N}-m\beta-\frac{\mu_E(S-1)(qP-m\nu)\pi}{PNS}\right].$$

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Here the maximum number r differs from the minimum number q by 1 if $\frac{m\nu}{P}$ is not an integer, and by 2 if $\frac{m\nu}{P}$ is an integer. However, if $\frac{m\nu}{P}$ is known to be an integer in some expression, then it is possible to assume that in this case also the minimum number q is greater by 1 than the number r and that either r or q is equal to $\frac{m\nu}{P}$. Although we are thereby formally introducing into the spectrum, it would seem, one additional new frequency (equal to zero), as a matter of fact we are not adding anything, inasmuch as the amplitude of this frequency is taken to be equal to zero in accordance with formula (75). Let us introduce a new ordinal number ν in the addition and write expression (75) in the form

$$\begin{aligned} F(t) = {} & \frac{E}{NS} + \sum_{q=1}^{\infty} \frac{2E}{q\pi} J_0 \left[\frac{\mu_E (S-1) q\pi}{NS} \right] \sin \frac{q\pi}{NS} \times \\ & \times \cos \left[2qP\pi t - \frac{q\pi}{N} - \frac{\mu_E (S-1) q\pi}{NS} \right] + \\ & + \sum_{u=1}^{\infty} \sum_{m=1}^{\infty} \frac{2EP}{(uP+m\nu)\pi} J_m \left[\frac{\mu_E (S-1)(uP+m\nu)\pi}{PNS} \right] \sin \frac{(uP+m\nu)\pi}{PNS} \times \\ & \times \cos \left[2(uP+m\nu)\pi t - \frac{u\pi}{N} + m\beta + m\pi - \frac{\mu_E (S-1)(uP+m\nu)\pi}{PNS} \right] + \\ & + \sum_{\nu=0}^{\infty} \sum_{m=1}^{\infty} \frac{2EP}{(\nu P-m\nu)\pi} J_m \left[\frac{\mu_E (S-1)(\nu P-m\nu)\pi}{PNS} \right] \sin \frac{(\nu P-m\nu)\pi}{PNS} \times \\ & \times \cos \left[2(\nu P-m\nu)\pi t - \frac{\nu\pi}{N} - m\beta - \frac{\mu_E (S-1)(\nu P-m\nu)\pi}{PNS} \right]. \end{aligned} \qquad (76)$$

Here ν is any positive integer or zero. It is easily seen that if ν is greater than $\frac{m\nu}{P}$, the last double sum of formula (76) is reduced to the last double sum of formula (75); and that if ν is less than $\frac{m\nu}{P}$, the last double sum of formula (76) is reduced to the last double sum but one of formula (75) and, finally, that if ν equals $\frac{m\nu}{P}$ the last double sum of formula (76) is reduced to zero.

Let us combine the first sum with the first double sum and rewrite expression (76) for the last time in the final form

$$\begin{aligned} F(t) = {} & \frac{E}{NS} + \sum_{u=1}^{\infty} \sum_{m=0}^{\infty} \frac{2EP}{(uP+m\nu)\pi} J_m \left[\frac{\mu_E (S-1)(uP+m\nu)\pi}{PNS} \right] \sin \frac{(uP+m\nu)\pi}{PNS} \times \\ & \times \cos \left[2(uP+m\nu)\pi t + m\pi - \frac{u\pi}{N} + m\beta - \frac{\mu_E (S-1)(uP+m\nu)\pi}{PNS} \right] + \\ & + \sum_{\nu=0}^{\infty} \sum_{m=1}^{\infty} \frac{2EP}{(\nu P-m\nu)\pi} J_m \left[\frac{\mu_E (S-1)(\nu P-m\nu)\pi}{PNS} \right] \sin \frac{(\nu P-m\nu)\pi}{PNS} \times \\ & \times \cos \left[2(\nu P-m\nu)\pi t - \frac{\nu\pi}{N} - m\beta - \frac{\mu_E (S-1)(\nu P-m\nu)\pi}{PNS} \right]. \end{aligned}$$

(77)

This is also formula (23).

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While this article was being proofread, still another article on frequency spectra appeared:

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[Figures follow.]

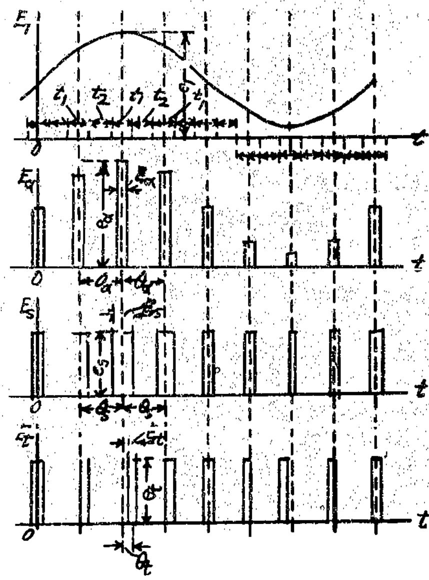

Figure 1. Various Types of Pulse Modulation: E_1 -- form of signal, depending on transmission, E_a -- amplitude -- pulse modulation; E_s -- time -- pulse modulation; E_ϕ phase pulse modulation.

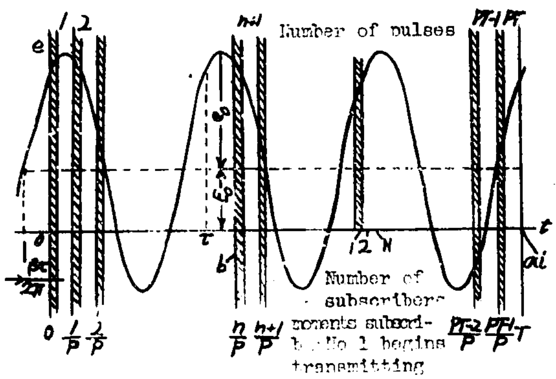

Figure 2. Distribution of Transmission Time Between Individual Subscribers.

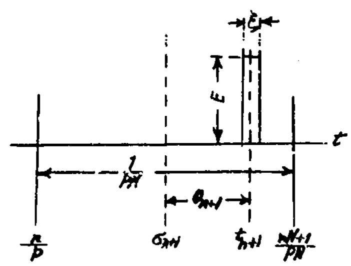

Figure 3. Pulse Number n + 1

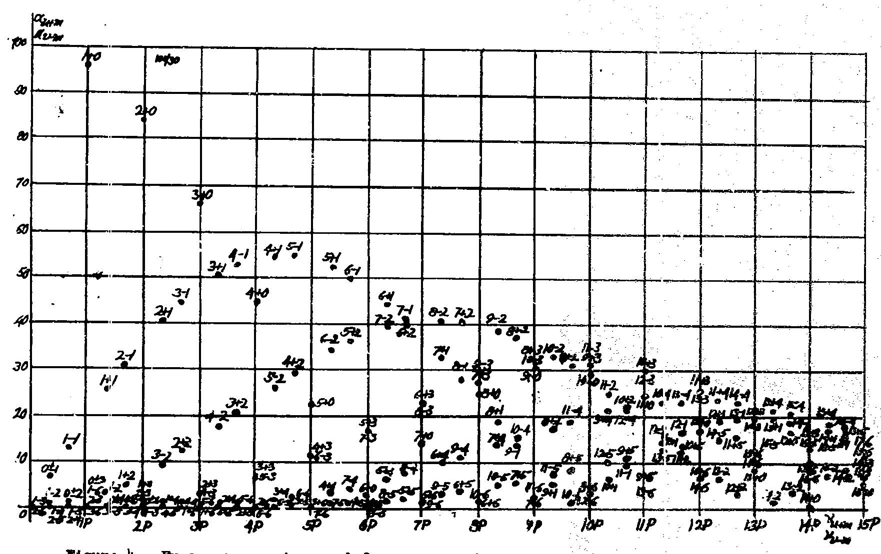

Figure 4. Frequency spectrum and frequency amplitudes up to the fifteenth order of the fundamental overtone and side frequency components (up to the sixth order inclusively) with numerical values (31), (32), (33), and (35), if there is only one modulating frequency (34); the figures in the diagram represent the indexes u + m and v − m with a from the formulas (26) and (29)

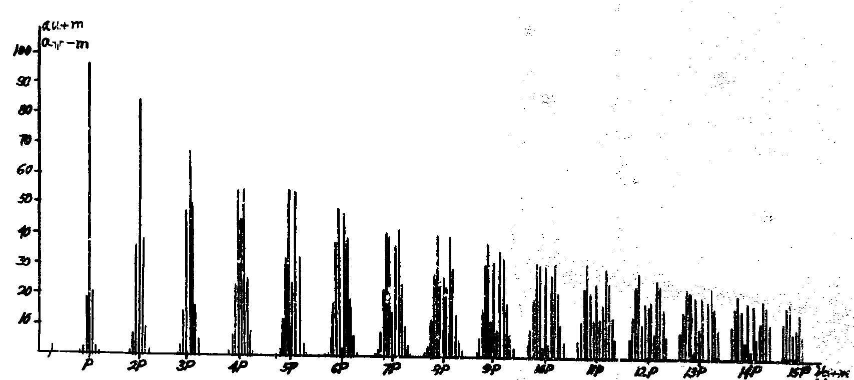

Figure 5. Spectrum of frequencies and their amplitudes up to the fifteenth order of the fundamental overtone, and side components (up to the sixth order inclusive) with the numerical values (31), (32), (33), and (35), if there is only one modulating frequency (36)

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